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ON METHODS OF SEARCHING OF POLYNOMIAL CONSERVATION LAWS FOR SOME MULTIDIMENSIONAL EVOLUTION EQUATIONS USING GENERALIZED EULER OPERATOR AND THEIR IMPLEMENTATION BY TOOLS OF INFORMATION TECHNOLOGIES

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Abstract

The work is devoted to research of Polynomial Conservation Laws (PCL) for one family of evolution equations. For polynomial differential operator is found one necessary and sufficient condition when it is a density of PCL. Method of undetermined coefficients is also presented, which allows to determine coefficients of a polynomial differential operator when it is a density of PCL. Finally, shortly is reviewed software package, which is designed to implement presented methods.

Key words and phrases: Korteweg-de Vries equation, Conservation law, Euler operator, System of linear equations.

AMS subject classification: 65-04, 35Q53.

1. Introduction

In the paper there is considered opened subset Ω of space R^n, where R denotes field of real numbers and a set of infinitely differentiable functions on $\Omega \times]0, t_0[$, where t_0 is a fixed positive number, on $C^\infty(\Omega \times]0, t_0[)$ - the following set of differential operators

$$V = \left\{ \prod_{i=1}^{m} \frac{\partial^{p^i} \bullet}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 \dots \partial^{p_n^i} x_n} \right. \left| p^i = \sum_{k=1}^{n} p_k^i, p_k^i, p^i \in N_0, \quad i = 1, \dots, m, \quad k = 1, \dots, n \right\}$$

$$W = \left\{ \sum_{i=1}^{M} \mu_i \prod_{j=1}^{m_i} \frac{\partial^{p^{(i,j)}} \bullet}{\partial^{p_1^{(i,j)}} x_1 \partial^{p_2^{(i,j)}} x_2 \dots \partial^{p_n^{(i,j)}} x_n} \right| \mu_i \in R, \qquad (1.1)$$
$$\left. p(i,j) = \sum_{k=1}^{n} p_k(i,j), \; p_k(i,j), p(i,j) \in N_0, \; i = 1, \dots M, \; j = 1, \dots m_i \right\}.$$

The paper is devoted to the methods of searching PCL for evolution equations of the following type

$$\frac{\partial u}{\partial t} + L[u] = 0, \qquad (1.2)$$

The similar problem for Korteweg-de Vries Equation (KdV) is investigated in [1],[2]. KdV equation has the following form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

It is a particular case of (1.2), when $n = 1$. The results are presented in more general form for larger class of equations in this paper than in [1],[2].

We have also presented methods of implementation of the theory using Information Technologies. Appropriate software is designed in Borland Delphi 6.0, which also is shortly described.

2. About One Necessary and Sufficient Condition of Existence of Polynomial Conservation Law

Let us assume, that S_1 and S_2 elements of W are equivalent, if exists the sequence $\{R_i \in W \,|i = 1,\ldots,n\}$ satisfying the condition

$$S_1 - S_2 = \sum_{i=1}^{n} \frac{\partial R_i}{\partial x_i}.$$

In order to denote that S_1 and S_2 operators are equivalent, usually is used the designation $S_1 \sim S_2$ [3]. The Euler operator has the following form

$$J\left(T\left[u\right]\right) = \sum_{p \in N_0} (-1)^p \frac{\partial^p}{\partial x^p}\left(\frac{\partial T\left[u\right]}{\partial\left(\frac{\partial^p u}{\partial x^p}\right)}\right), \tag{2.1}$$

where $x \in R$. We consider the following generalization of (2.1) in n-dimensional space case and call it Generalized Euler Operator (GEO):

$$J\left(T\left[u\right]\right) = \sum_{p_i \in N_0, p=\sum_{i=1}^{n} p_i} (-1)^p \frac{\partial^p}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\left(\frac{\partial T\left[u\right]}{\partial\left(\frac{\partial^p u}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\right)}\right). \tag{2.2}$$

Lemma 1. *$S \sim 0$ if and only if $J(S) = 0$, where $S \in W$ and J denotes GEO.*

The proof of this Lemma in one-dimensional space case is presented in [2]. The proof for GEO is quite sophisticated and is beyond the scope of this paper.

Let u be the solution of (1.2). In one dimensional space case $T \in W$ is considered as a density of equation (1.1), if exists $X \in W$ such, that

$$\frac{\partial T\left[u\right]}{\partial t} + \frac{\partial X\left[u\right]}{\partial x} = 0. \tag{2.3}$$

Equality (2.3) is called a Polynomial Conservation Law (PLC) of (1.2).

In n-dimensional space case we call $T \in W$ a density of equation (1.1), if exists a sequence of operators $\{X \in W_i \,|i = 1,\ldots,n\}$ satisfying the equality

$$\frac{\partial T\left[u\right]}{\partial t} + \sum_{i=1}^{n} \frac{\partial X_i\left[u\right]}{\partial x_i} = 0. \tag{2.4}$$

We call (2.4) a Polynomial Conservation Law (PCL) of (1.2) [4],[5].

Lemma 2. *Let $S_1, S_2 \in W$. Then the following equality is correct* $\frac{\partial^p S_1}{\partial x_1^{p_1}\partial x_2^{p_2}...\partial x_n^{p_n}} S_2 \sim (-1)\, S_1 \frac{\partial^p S_2}{\partial x_1^{p_1}\partial x_2^{p_2}...\partial x_n^{p_n}}$

Proof.

$$\frac{\partial^p S_1}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}} S_2 = \frac{\partial}{\partial x_1}\left[\frac{\partial^{p-1} S_1}{\partial x_1^{p_1-1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}} S_2\right] -$$

$$-\frac{\partial^{p-1} S_1}{\partial x_1^{p_1-1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial S_2}{\partial x_1} \sim \frac{\partial^{p-1} S_1}{\partial x_1^{p_1-1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial S_2}{\partial x_1} \sim$$

$$\sim (-1)^{p_1}\frac{\partial^{p-p_1} S_1}{\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial^{p_1} S_2}{\partial x_1} \sim \cdots \sim (-1)^{\sum_{i=1}^{n} p_i} S_1 \frac{\partial^p S_2}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}.$$

Lemma 3. *If $T \in W$ and u is the solution of (1.2), then exists a sequence of operators $\{Z_i \in W \,|\, i = 1, \ldots, n\}$ satisfying the following condition:*

$$\frac{\partial T[u]}{\partial t} = -J(T[u])\, L[u] + \sum_{i=1}^{n}\frac{\partial Z_i[u]}{\partial x_i},$$

where J is GEO.

Proof. Let

$$A[u] = \prod_{j=1}^{m}\frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n}.$$

Due to definition of GEO

$$\frac{\partial A[u]}{\partial t} = \frac{\partial}{\partial t}\prod_{j=1}^{m}\frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n} =$$

$$= \sum_{i=1}^{m}\left[\left(\prod_{j=1}^{i-1}\frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n}\right)\times\right.$$

$$\left.\times\frac{\partial^{p^i+1} u}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 \ldots \partial^{p_n^i} x_n \partial t}\left(\prod_{j=i+1}^{m}\frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n}\right)\right]. \tag{2.5}$$

Let us consider the following derivatives of (1.2):

$$\frac{\partial^{p^{j+1}} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n \partial t} + \frac{\partial^{p^{j+1}} L[u]}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 \ldots \partial^{p_n^j} x_n} = 0, \quad j = 1, \ldots, n. \tag{2.6}$$

According to (2.5) and (2.6)

$$\frac{\partial A[u]}{\partial t} = -\sum_{i=1}^{m}\frac{\partial A[u]}{\partial\left(\frac{\partial^{p^i} u}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 \ldots \partial^{p_n^i} x_n}\right)}\frac{\partial^{p^i} L[u]}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 \ldots \partial^{p_n^i} x_n}. \tag{2.7}$$

$$\frac{\partial A[u]}{\partial t} \sim \sum_{i=1}^{n}(-1)^{p^i}\frac{\partial^{p^i}}{\partial^{p_1^i}x_1\partial^{p_2^i}x_2\ldots\partial^{p_n^i}x_n}\cdot\frac{\partial A[u]}{\partial\left(\frac{\partial^{p^i}u}{\partial^{p_1^i}x_1\partial^{p_2^i}x_2\ldots\partial^{p_n^i}x_n}\right)}L[u] =$$

$$= -J(A[u])\,L[u]. \tag{2.8}$$

Let

$$T = \sum_{i=1}^{N}\lambda_i A_i,$$

where $A_i \in V, \lambda_i \in R : i = 1, ..., N.$

$$\frac{\partial T[u]}{\partial t} = \sum_{i=1}^{N}\lambda_i\frac{\partial A_i[u]}{\partial t} = -\sum_{i=1}^{N}\lambda_i J(A_i[u])\,L[u] + \sum_{i=1}^{N}\sum_{j=1}^{n}\lambda_i\frac{\partial Y_j^i[u]}{\partial x_j} =$$

$$= -J(T[u])\,L[u] + \sum_{j=1}^{n}\lambda_i\frac{\partial Z_j[u]}{\partial x_j},$$

where

$$Z_j[u] = \sum_{i=1}^{N}\lambda_i Y_j^i[u], \quad Y_j^i \in W, \quad i = 1, ..., N, \quad j = 1, ..., n,$$

what proves Lemma 3.

Now let us prove the following theorem.

Theorem. $T \in W$ *is a density of some PCL of (1.2) if and only if* $J(J(T)\cdot L) = 0$.

Proof. Necessity: Let T be a density of (1.2). Then exist $X_i \in W$, $i = 1, ..., m$, which satisfy equality

$$\frac{\partial T[u]}{\partial t} + \sum_{i=1}^{n}\frac{\partial X_i[u]}{\partial x_i} = 0. \tag{2.9}$$

Due to Lemma 3, exist $\{Z_i \in W \,|i = 1, \ldots, n\}$ satisfying condition

$$\frac{\partial T[u]}{\partial t} = -J(T[u])\cdot L[u] + \sum_{i=1}^{n}\frac{\partial Z_i[u]}{\partial x_i}. \tag{2.10}$$

From (2.9) and (2.10) we can conclude, that

$$J(T[u])\cdot L[u] = \sum_{i=1}^{n}\frac{\partial(X_i + Z_i)[u]}{\partial x_i}. \tag{2.11}$$

It is evident, that

$$\sum_{i=1}^{n}\frac{\partial(X_i + Z_i)[u]}{\partial x_i} \sim 0. \tag{2.12}$$

Due to (2.12) and Lemma 1

$$J\left(\sum_{i=1}^{n}\frac{\partial\left(X_i+Z_i\right)[u]}{\partial x_i}\right)=0. \tag{2.13}$$

From (2.11) and (2.13) we obtain

$$J\left(J\left(T\right)\cdot L\right)=0,$$

that proves necessity.

Sufficiency: Let $J\left(J\left(T\right)\cdot L\right)=0$. Then, due to Lemma 1, $J\left(T\right)\cdot L\sim 0$, that means the existence $\{Y_i\in W\,|i=1,\ldots,n\}$ satisfying condition

$$J\left(T\right)\cdot L=\sum_{i=1}^{n}\frac{\partial Y_i}{\partial x_i}. \tag{2.14}$$

Let u be a solution of (1.2). Then, due to Lemma, exist such $\{Z_i\in W\,|i=1,\ldots,n\}$, that

$$\frac{\partial T\,[u]}{\partial t}=-J\left(T\,[u]\right)L\,[u]+\sum_{i=1}^{n}\frac{\partial Z_i\,[u]}{\partial x_i}. \tag{2.15}$$

From (2.14) and (2.15) we can conclude, that

$$\frac{\partial T\,[u]}{\partial t}+\sum_{i=1}^{n}\frac{\partial\left(Y_i\,[u]-Z_i\,[u]\right)}{\partial x_i}=0,$$

what means, that T is a density of (1.2).

If $J\left(T\right)=0$, then T operator is called a trivial density [1],[2], as it is a density for all equations of type (1.2). Indeed, as $J\left(J\left(T\right)\cdot L\right)=0$, T is a density of (1.2) for any $L\in W$.

3. Method of Undetermined Coefficients

Let

$$T=\sum_{i=1}^{n}\lambda_i A_i, \tag{3.1}$$

where $A_i\in V$, $\lambda_i\in R$, $i=1,\ldots,n$.

We search coefficients $(\lambda_i)_{i=0}^{N}$, when T is a density of (1.2).

Due to Theorem, T is a density of (1.2), if and only if

$$J\left(L\cdot J\left(T\right)\right)=0. \tag{3.2}$$

From (3.1) and (3.2) we obtain, that

$$\sum_{i=1}^{N}\lambda_i J\left(L\cdot J\left(A_i\right)\right)=0. \tag{3.3}$$

Let

$$J\left(L\cdot J\left(A_i\right)\right)=\sum^{N_i}\mu_j^i B_j^i,\quad \mu_j^i\in R,\quad B_j^i\in V,\quad j=1,\ldots,N_i,\quad i=1,\ldots,N.$$

We construct

$$X = \left\{ B_j^i \mid \quad i = 1, \ldots, N, \quad j = 1, \ldots, N_i \right\}$$

and renumber elements of X

$$X = \{ C_j \mid \quad i = 1, \ldots, M \} .$$

Let

$$J\left(L \cdot J\left(A_i\right)\right) = \sum_{j=1}^{M} \eta_j^i C_j, \quad j = 1, \ldots, N. \tag{3.4}$$

Using (3.4), equality (3.3) takes the following form

$$\sum_{j=1}^{M} \left(\sum_{i=1}^{N} \eta_j^i \lambda_i \right) C_j = 0. \tag{3.5}$$

Equality (3.5) takes place if and only if

$$\sum_{i=1}^{N} \eta_j^i \lambda_i = 0, \quad j = 1, \ldots, M. \tag{3.6}$$

Obtained condition (3.6) is a system of linear equations and if $(\lambda_i)_{i=1}^N$ is the solution of (3.6), then T is the density of (1.2). It means, that the searching of $(\lambda_i)_{i=1}^N$ coefficients, when T is a density, leads to solving the system of linear equations (3.6).

As an example let us consider one generalized form of KdV Equation

$$\frac{\partial u}{\partial t} + u \left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \right) + \left(\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} \right) = 0 \tag{3.7}$$

and search its density in the following form

$$T = \lambda_1 u^3 + \lambda_2 \left(\frac{\partial u}{\partial x_1} \right)^2 + \lambda_3 \left(\frac{\partial u}{\partial x_2} \right)^2 + \lambda_4 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2}. \tag{3.8}$$

By the definition of GEO,

$$J\left(T\right) = 3\lambda_1 u^2 - 2\lambda_2 \frac{\partial^2 u}{\partial x_1^2} - 2\lambda_3 \frac{\partial^2 u}{\partial x_2^2} - 2\lambda_4 \frac{\partial^2 u}{\partial x_1 \partial x_2}$$

and equation (3.5) takes the following form

$$-18\lambda_1 \left(\frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_2^2} \right) - 2\lambda_2 \left(3 \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1^2} \frac{\partial u}{\partial x_2} \right) -$$

$$-2\lambda_3 \left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial u}{\partial x_2} + 3 \frac{\partial^2 u}{\partial x_2^2} \frac{\partial u}{\partial x_2} \right) - 2\lambda_4 \left(2 \frac{\partial^2 u}{\partial x_1^2} \frac{\partial u}{\partial x_2} + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial u}{\partial x_2} \right) = 0. \tag{3.9}$$

Let

$$X = \left\{ \frac{\partial^2 u}{\partial x_1^2} \frac{\partial u}{\partial x_1}, \frac{\partial^2 u}{\partial x_1} \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_2^2} \frac{\partial u}{\partial x_2} \right\}$$

and number its elements. From (3.9) we obtain the following system of linear equations

$$\begin{cases} -18\lambda_1 - 6\lambda_2 = 0 \\ -2\lambda_2 - 2\lambda_4 = 0 \\ -4\lambda_3 - 4\lambda_4 = 0 \\ -18\lambda_1 - 6\lambda_3 = 0 \end{cases} \tag{3.10}$$

and its solution is

$$\begin{cases} \lambda_1 = \frac{1}{3}C \\ \lambda_2 = -C \\ \lambda_3 = -C \\ \lambda_4 = C \end{cases} \tag{3.11}$$

where $C \in R$, so

$$T = \frac{1}{3}Cu^3 - C\left(\frac{\partial u}{\partial x_1}\right)^2 - C\left(\frac{\partial u}{\partial x_2}\right)^2 + C\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}.$$

4. About Software Package for Searching the PLC of Evolution Equation

The software package for searching PCL of evolution equations consists of three different program. First one is an editor of elements of W and it looks like Fig. 1:

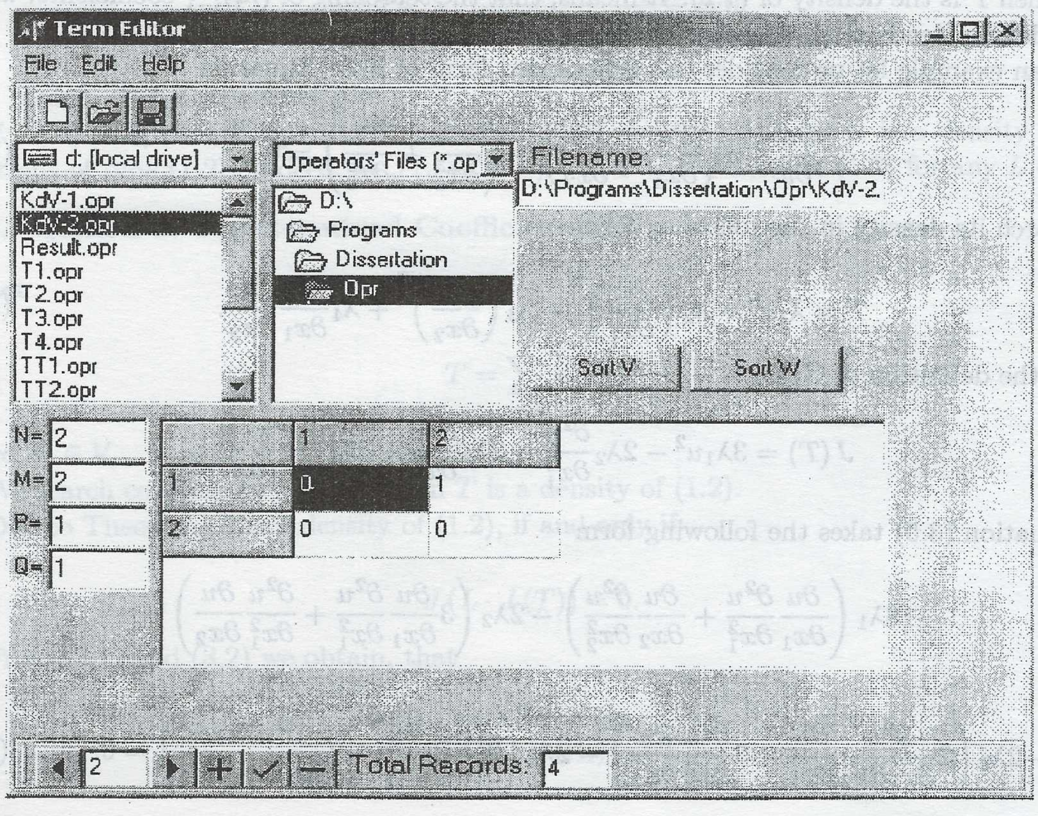

Fig.1

We have designed specific file format for elements of W. This editor allows to create and edit them. Each term of the element of W is represented with matrix of integer numbers. Its coefficients are pairs of integer numbers with meanings of numerator and denominator (we do not consider the case when coefficients are not rational). Built-in sorting procedures guarantee that the elements of V and W are presented in only one way.

In the second program are implemented operations on elements of W (Fig. 2).

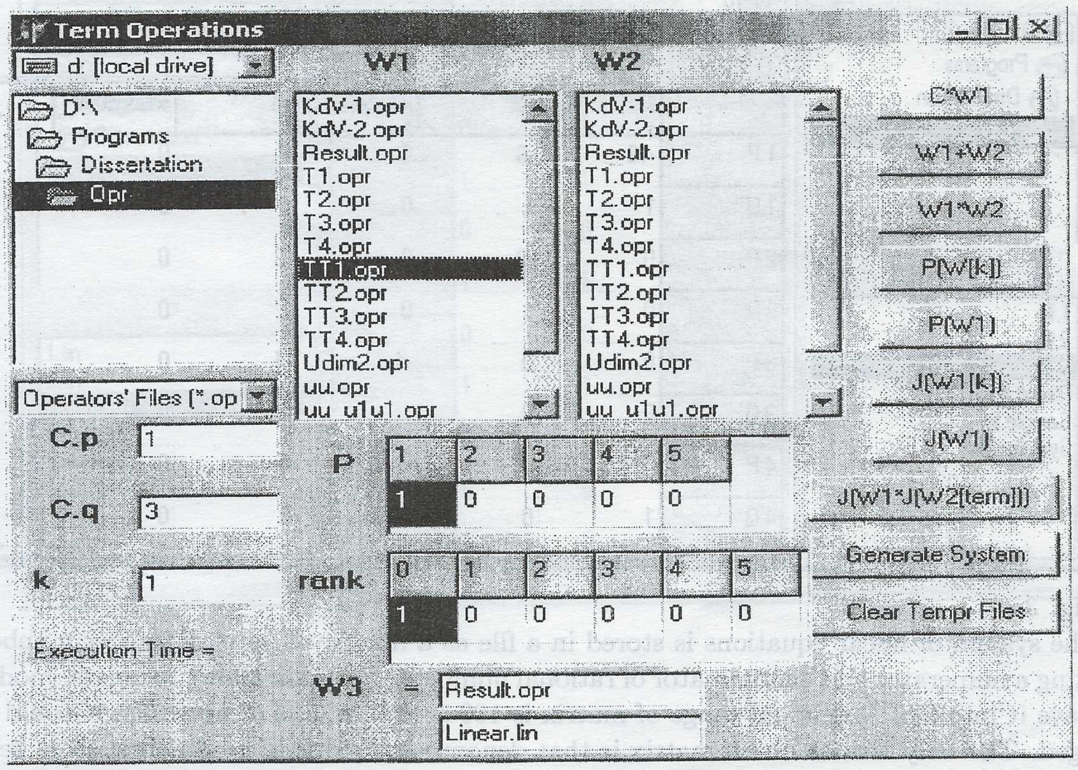

Fig. 2

Arguments of implemented functions and procedures are elements of W, which are selected from $W1$ and $W2$ list boxes, whereas results are stored in $W3$.

One of the procedures generates the system of linear equations for

$$\frac{\partial u}{\partial t} + W_1\,[u] = 0$$

equation and W_2 density, i.e. system (3.6), assuming that $L = W_1$ and $T = W_2$. The system of linear equations is stored in a specific file format.

The third program solves the system of linear equations (Fig. 3).

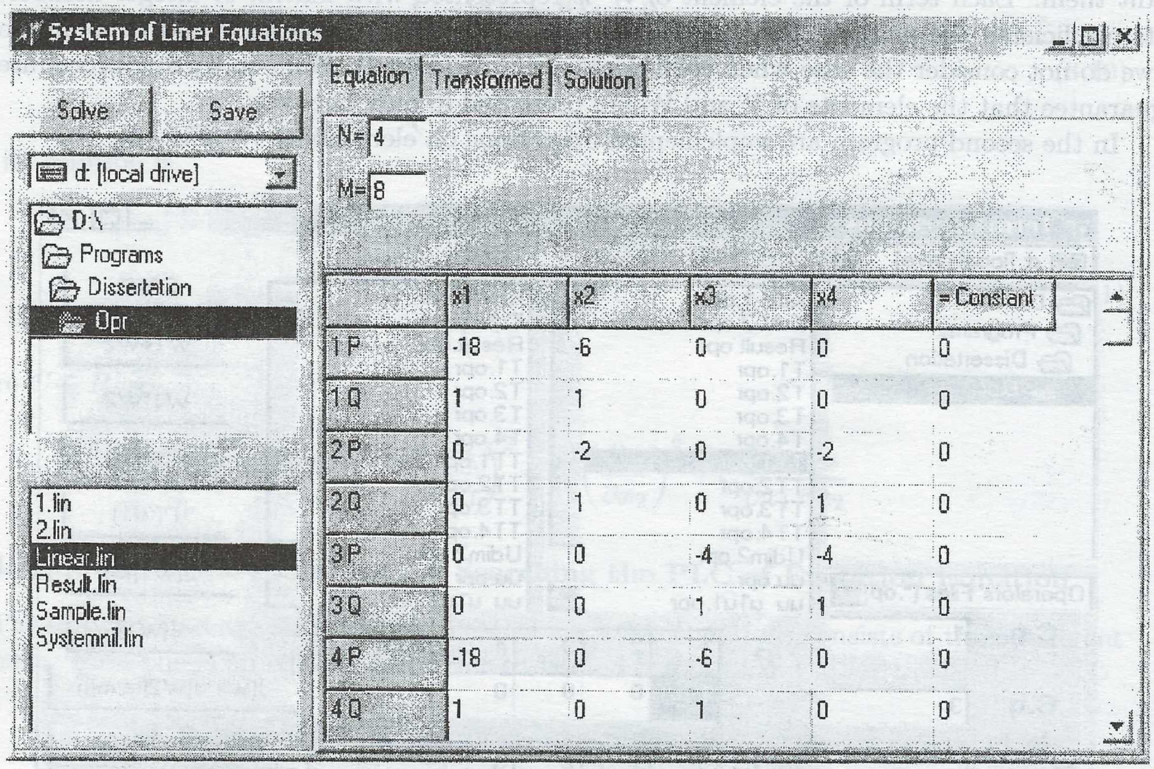

Fig. 3

The system of linear equations is stored in a file as a matrix of pairs of integer numbers denoting a numerator and denominator of rational numbers. Program allows to create, modify and save it into file. Maximum range of matrix is 100×100. In case of necessity, it could be enlarged. The only limitation on matrix is that denominators should be different from zero. The number of variables could exceed the number of equations and vice versa.

By Clicking on button < *Solve* >, program solves the system. On Page < *transformed* > is presented a diagonal form of matrix after transformation (Fig. 4).

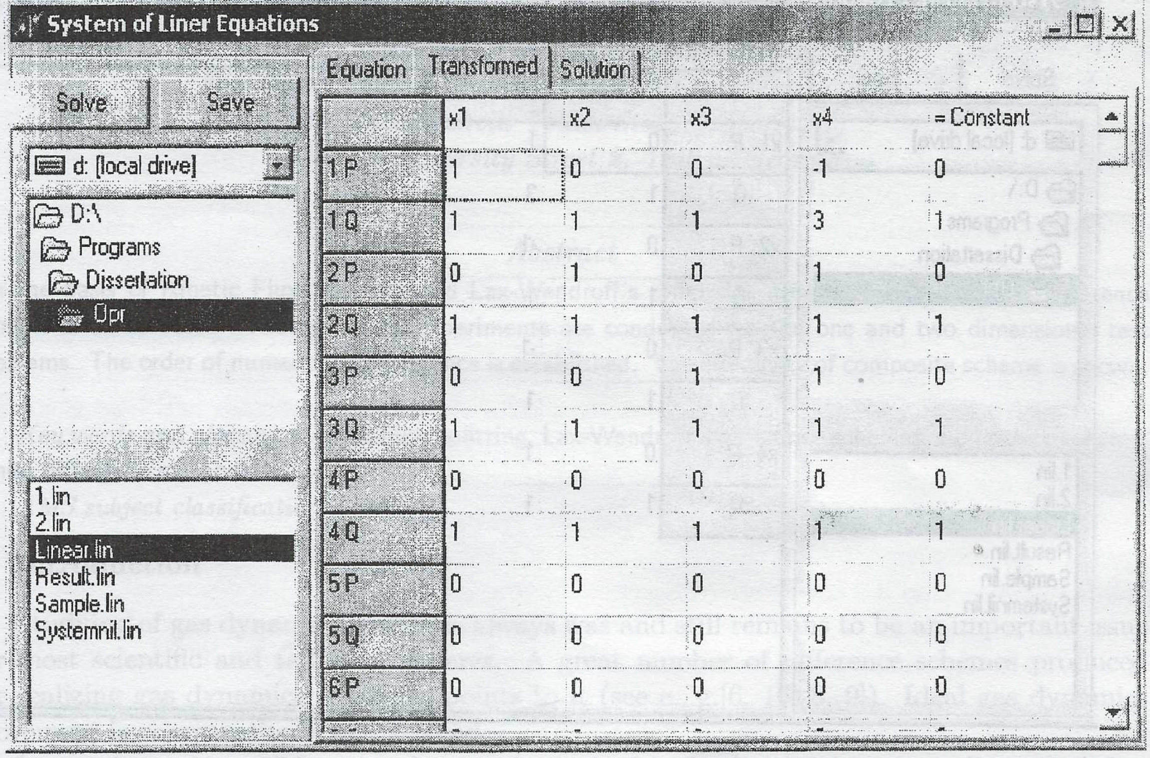

Fig. 4

On page < *Solution* > is presented solution of the system (Fig. 5).

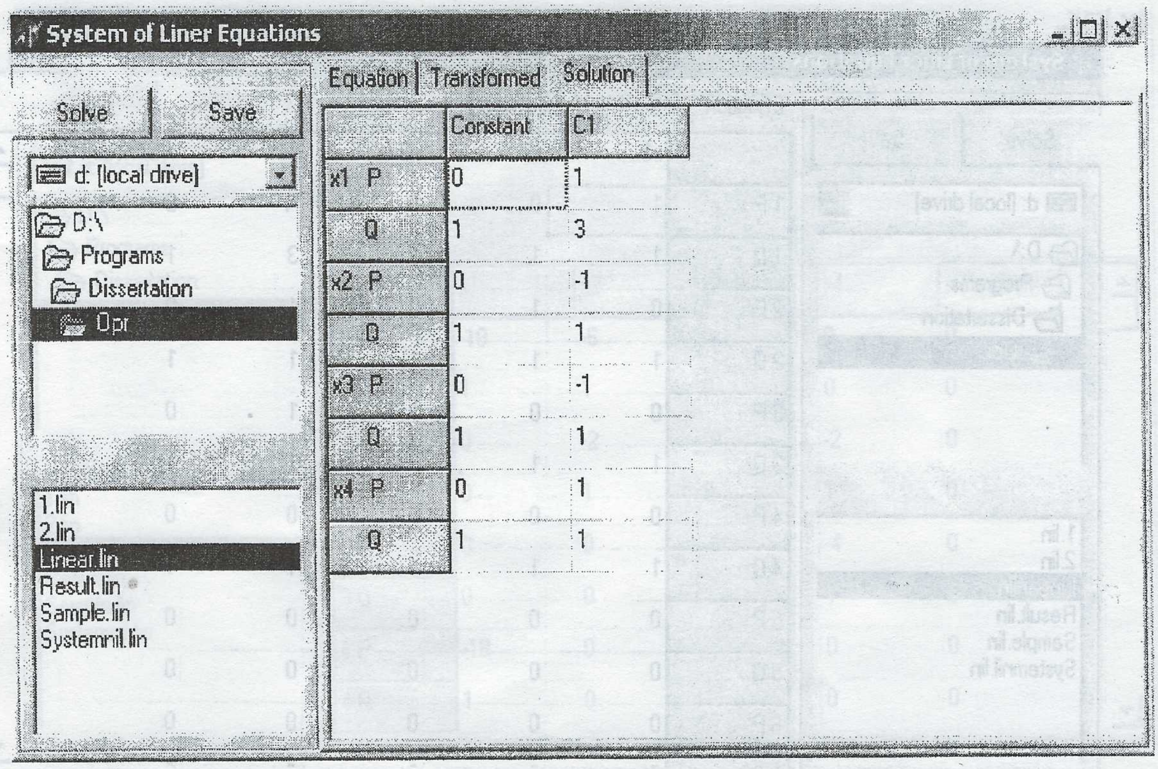

Fig. 5

Figures 3,4,5 illustrate, that the solution of system (3.10) is (3.11).

Thus, we have proved that $T = \sum_{i=1}^{N} \lambda_i A_i \in W$ is a density of (1.2) if and only if $J(J(T) \cdot L) = 0$, where J is Generated Euler Operator defined by (2.2). Developed method of undetermined coefficients for searching $(\lambda_i)_{i=0}^{N}$, when T is a density of (1.2), which leads to solving of the system of linear equations (3.6). Reviewed software package, which creates and edits elements of W, generates system of linear equations (3.6) and solves it.

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