ISSN 1512-0066

თბილისის სახელმწიფო უნივერსიტეტი • TBILISI STATE UNIVERSITY

ი. ვეკუას სახელობის
გამოყენებითი მათემატიკის ინსტიტუტის სემინარის
გაფართოებული სხდომების

მოხსენებები

ტ. 12, № 1-3
1997

REPORTS

OF ENLARGED SESSIONS OF THE SEMINAR OF
I. VEKUA INSTITUTE OF APPLIED MATHEMATICS

VOL. 12 , № 1-3
1997

Reports of Enlarged Session of the
Seminar of I. Vekua Institute
of Applied Mathematics
Volume 12, № 3, 1997

ON SEARCHING OF POLYNOMIAL CONSERVATION LAWS FOR SOME MULTIDIMENSIONAL EVOLUTION EQUATIONS USING GENERALIZED EULER OPERATOR

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1. INTRODUCTION. The subject of our research is multidimensional evolution equation

$$\frac{\partial(x_1,x_2,...x_n,t)}{\partial t}+Lu(x_1,x_2,...x_n,t)=0, \tag{1}$$

where

$$Lu=\sum_{i=1}^{M}(\mu_i\prod_{j=1}^{M}\frac{\partial^{r(i,j)u}}{\partial^{r_1(i,j)}x_1\partial^{r_2(i,j)}x_2...\partial^{r_n(i,j)}x_n},$$

$$M,m_i\in N,r(i,j),r_k(i,j)\in N_0,r(i,j)=\sum_{k=1}^{n}r_k(i,j),$$

namely Polynomial Conservation Laws (P.C.L.) of (1), which are often used to prove existence and uniqueness of the solution.

The similar problem for Korteweg - de Vries equation is treated in papers [3],[4]. This paper is an attempt to enrich methodology of searching P.C.L. in larger class of equations.

We treat this problem in the special class of operators. First we define this set and several concepts associated with it. Then we present three theorems and as a demonstration of the method consider the simplest Conservation Law for a linear equation.

2. **Definitions.** Let set of operators $\sum_{i=1}^{M}\mu_i\prod_{j=1}^{m_i}\frac{\partial^{p(i,j)}*}{\partial^{p_1(i,j)}x_1\partial^{p_2(i,j)}x_2...\partial^{p_n(i,j)}x_n}$ be W. Let $S_1,S_2\in W$. We will assume that $S_1\sim S_2$ if $\exists A_i\in W\ i=\overline{1,n}$ that

$$S_1-S_2=\sum_{i=1}^{n}\frac{\partial A_I}{\partial x_i}. \tag{2}$$

We will call $T\in W$ the density of (1) if there exists such $X_i\in W,\ i=\overline{1,m}$ that

$$\frac{\partial T[u]}{\partial t}+\sum_{i=1}^{n}\frac{\partial X_i[u]}{\partial x_i}=0, \tag{3}$$

we call (3) the Polynomial Conservation Law of (1).

It is easy to see that if $T_1,T_2,...,T_N$ are densities then $\sum_{i=1}^{N}\lambda_iT_i$ is density also.

On W we define rank, Projector and Euler Operator as follows:

$$a)rank\left(\prod_{i=1}^{m}\frac{\partial^{p^i}*}{\partial^{p_1^i}x_1\partial^{p_2^i}x_2...\partial^{p_n^i}x_n}\right)=]m,\sum_{i=1}^{m}p_1^i,...,\sum_{i=1}^{m}p_n^i[, \tag{4}$$

$$b) Prj(]m, m_1, m_2, ..., m_n[, \sum_{i=1}^{N} \bar{\lambda}_i R_i,$$

where

$$\bar{\lambda}_i = \begin{cases} \lambda_i \; if \; rank \; R_i =]m, m_1, ..., m_n[\\ 0 \; if \; rank \; R_i \neq]m, m_1, ..., m_n[\end{cases} \tag{5}$$

$$c) J(S[u]) = \sum_{p_1 \in N_0, p = \sum_{i=1}^{n} p_i} (-1)^p \frac{\partial^p}{\partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}} \left(\frac{\partial S[u]}{\partial \left(\frac{\partial^p u}{\partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}} \right)} \right). \tag{6}$$

3. **Theorems.** *Let us prove one equivalence:*

$$\frac{\partial^p A}{\partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}} B \sim -\frac{\partial^{p-1} A}{\partial x_1^{p_1 - 1} \partial x_2^{p_2} ... \partial x_n^{p_n}} \frac{\partial B}{\partial x_1} \sim (-1)^{p_1} \frac{\partial^{p-p_1} A}{\partial x_2^{p_2} ... \partial x_n^{p_n}} \frac{\partial^{p_1} B}{\partial x_1^{p_1}} \sim ...$$

$$... \sim (-1)^p A \frac{\partial^p B}{\partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}}.$$

Statement. If $S \in W$, then $S \sim 0$ if and only if $J(S) = 0$.
Theorem 1. *T is the density of (1) if and only if $J(J(T)L) = 0$.*
Necessity.

$$\frac{\partial}{\partial t} \prod_{j=1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} =$$

$$\sum_{i=1}^{m} \left[\left(\prod_{j=1}^{i=1} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right) \frac{\partial^{p^i + 1} u}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 ... \partial^{p_n^i} x_n \partial t} \left(\prod_{j=i+1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right) \right] =$$

$$\sum_{i=1}^{m} \left[\frac{\partial \left(\prod_{j=1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right)}{\partial \left(\frac{\partial^{p^i} u}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 ... \partial^{p_n^i} x_n} \right)} \frac{\partial^{p^i}}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 ... \partial^{p_n^i} x_n} (Lu) \right] \sim$$

$$\sim -\sum_{i=1}^{m} \left[(-1)^{p^i} \frac{\partial^{p^i}}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \left[\frac{\partial \left(\prod_{j=1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right)}{\partial \left(\frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right)} \right] Lu \right] =$$

$$-J \left(\prod_{j=1}^{m} \frac{\partial^{p^i} u}{\partial^{p_1^i} x_1 \partial^{p_2^i} x_2 ... \partial^{p_n^i} x_n} \right) Lu,$$

so

$$\frac{\partial}{\partial t} \prod_{j=1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} = -J \left(\prod_{j=1}^{m} \frac{\partial^{p^j} u}{\partial^{p_1^j} x_1 \partial^{p_2^j} x_2 ... \partial^{p_n^j} x_n} \right) Lu + \sum_{i=1}^{n} \frac{\partial Y_i}{\partial x_i} \tag{7}$$

and as Euler operator is linear operator

$$\frac{\partial T[u]}{\partial t} = -J(T[u]) L[u] + \sum_{i=1}^{n} \frac{\partial Z_i[u]}{\partial x_i} \tag{8}$$

$$so\ J(T)L = \sum_{i=1}^{n} \frac{\partial(X_i + Z_i)}{\partial x_i}, \tag{9}$$

Due to statement $J(J(T)L) = 0$.

Sufficiency. As $J(J(T)L) = 0$, $J(T)L = \left(\sum_{i=1}^{n} \frac{\partial \bar{X}_i}{\partial x_i}\right.$. Due to (8) $\frac{\partial T}{\partial t} = \sum_{i=1}^{n} \frac{\partial(Z_i - \bar{X}_i)}{\partial x_i}$.

It is easy to see that if $J(T) = 0$ then $T \in W$ is density. Such densities we call trivial densities.

Theorem 2.

Let

$$L = \sum_{\bar{m}_0=\check{m}_0^1}^{\hat{m}_0^1} \sum_{\bar{m}_1=\check{m}_1^1}^{\hat{m}_1^1} \dots \sum_{\bar{m}_n=\check{m}_n^1}^{\hat{m}_n^1} Prj(]\bar{m}_0, \bar{m}_1, \dots \bar{m}_n, [, L). \tag{10}$$

$$T = \sum_{\bar{\bar{m}}_0=\check{m}_0^2}^{\hat{m}_0^2} \sum_{\bar{\bar{m}}_1=\check{m}_1^2}^{\hat{m}_1^2} \dots \sum_{\bar{\bar{m}}_n=\check{m}_n^2}^{\hat{m}_n^2} Prj(]\bar{m}_0, \bar{m}_1, \dots \bar{m}_n, [, L). \tag{11}$$

If T is the density of (1) *then $Prj(]\bar{\bar{m}}_0, \bar{\bar{m}}_1, \dots \bar{\bar{m}}_n[, T)$ is the density of the the following equation:*

$\frac{\partial u}{\partial t} + Prj(]\bar{m}_0, \bar{m}_1, \dots \bar{m}_n, [, L)[u] = 0$. *where next boundary condition takes place:*

$$\begin{cases} \check{m}_i^1 \le \bar{m}_i \le \hat{m}_i^1,\ \check{m}_i^2 \le \bar{\bar{m}}_i \le \hat{m}_i^2,\ i = \overline{0, n} \\ \sum_{i=0}^{n} \sigma_i(|\bar{m}_i - \check{m}_i^1| + |\bar{\bar{m}}_i - \check{m}_i^2|)(|\bar{m}_i - \hat{m}_i^1| + |\bar{\bar{m}}_i - \hat{m}_i^2|) = 0 \\ \sigma_i = \begin{cases} 0\ if\ (\check{m}_i^1 = \hat{m}_i^1) \text{ or } (\check{m}_i^2 = \hat{m}_i^2) \\ 1\ else \end{cases} \end{cases} \tag{12}$$

Note that proof is based on one statement which says that if $S \sim 0$ then $Prj(]m, m_1, m_2, \dots, m, [, S \sim 0$. We consider $LJ(T)$ operator, group its terms by ranks and using mentioned statement derive boundary condition (12).

Theorem 3.

Let $L = \sum_{i=1}^{M} \mu_i L_i$, *where rank* $L_i \ne$ *rank* L_j *if* $i \ne j$, *rank* $L_i =]a^i, a_1^i, a_2^i, \dots, a_n^i[$:

$T = \sum_{i=1}^{N} \lambda_i T_i$ *where rank* $T_i =$ *rank* T_j, *rank* $T_i =]b, b_1, b_2, \dots, b_n[$.

In these conditions T is density if and only if it is the density for each

$$\frac{\partial u}{\partial t} + \mu_i L_i[u] = 0, \quad i = \overline{1, M}. \tag{13}$$

Necessity.

As $\check{m}_i^2 = \hat{m}_i^2 \Rightarrow \sigma_i = 0\ i = \overline{1, n}$, so each $(a^i, a_1^i, a_2^i, \dots, a_n^i)$ is the solution of (12). Due to theorem 2 T is the density of (13).

Sufficiency.

Due to (13)

$J(T)L_j = \sum_{i=1}^{n} \frac{\partial X_j^i}{\partial x_j}$ so $J(T)L = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sum_{i=1}^{M} X_j^i \right)$ and due to Theorem 1 T is the density of (1).

4. **Demonstration of the Method.** Let us consider next operator

$$\frac{\partial^p *}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial^q *}{\partial x_1^{q_1}\partial x_2^{q_2}\ldots\partial x_n^{q_n}} \tag{14}$$

As $J\left(\frac{\partial^p *}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial^q *}{\partial x_1^{q_1}\partial x_2^{q_2}\ldots\partial x_n^{q_n}}\right)=((-1)^p+(-1)^q)\frac{\partial^{p+q}}{\partial x_1^{p_1+q_1}\partial x_2^{p_2+q_2}\ldots\partial x_n^{p_n+q_n}}$
it is trivial density if p and q have not the same evenness.

Let us find when (14) is a nontrivial density of next linear equation:

$$\frac{\partial u}{\partial t}+\mu\frac{\partial^r u}{\partial x_1^{r_1}\partial x_2^{r_2}\ldots\partial x_n^{r_n}}=0$$

$$J\left(J\left(\frac{\partial^p *}{\partial x_1^{p_1}\partial x_2^{p_2}\ldots\partial x_n^{p_n}}\frac{\partial^q *}{\partial x_1^{q_1}\partial x_2^{q_2}\ldots\partial x_n^{q_n}}\right)\frac{\partial^r *}{\partial x_1^{r_1}\partial x_2^{r_2}\ldots\partial x_n^{r_n}}\right)=$$

$$J\left(((-1)^p+(-1)^q)\frac{\partial^{p+q}}{\partial x_1^{p_1+q_1}\partial x_2^{p_2+q_2}\ldots\partial x_n^{p_n+q_n}}\frac{\partial^r}{\partial x_1^{r_1}\partial x_2^{r_2}\ldots\partial x_n^{r_n}}\right)=$$

$$[(-1)^p+(-1)^q][(-1)^{p+q}+(-1)^r]\frac{\partial^{p+q+r}}{\partial x_1^{p_1+q_1+r_1}\partial x_2^{p_2+q_2+r_2}\ldots\partial x_n^{p_n+q_n+r_n}}.$$

So (14) is a nontrivial density if and only if r is odd and p and q have the same evenness.

Now let us consider the following linear evolution equation:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{M}\mu_i\frac{\partial^{r^i} u}{\partial x_1^{r_1^i}\partial x_2^{r_2^i}\ldots\partial x_n^{r_n^i}}=0 \tag{15}$$

If (14) is the density of (15) then due to Theorem 3 it must be the density of following equations also:

$$\frac{\partial u}{\partial t}+\mu_i\frac{\partial^{r^i} u}{\partial x_1^{r_1^i}\partial x_2^{r_2^i}\ldots\partial x_n^{r_n^i}}=0 \quad i=\overline{1.M} \tag{16}$$

so we have the next

Result. (14) is the nontrivial density of linear equation (16) if and only if each r^i is odd and $p+q$ is even. So each

$$\sum_{i=1}^{N}\frac{\partial p^i *}{\partial x_1^{p_1^i}\partial x_2^{p_2^i}\ldots\partial x_n^{p_n^i}}\frac{\partial q^i *}{\partial x_1^{q_1^i}\partial x_2^{q_2^i}\ldots\partial x_n^{q_n^i}}$$

is the density of (16) if each r^i is odd.

As we see, using theorems 1,2,3 we simply derive results for linear equations, but real power of these theorems could be demonstrated on nonlinear equations, but it is out of scope of this paper.

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