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ON POLINOMIAL CONSERVATION LAWS FOR SOME EVOLUTION EQUATIONS

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INTRODUCTION

The aim of this paper is the construction of Polinomial Conservation Laws (P.C.L) for equation

$$\frac{\partial u(x,t)}{\partial t} + Lu(x,t) = 0, \quad where \;\; Lu = \sum_{i=1}^{p}\left(\mu_i \prod_{j=1}^{n_i} \frac{\partial^{r_i^j} u}{\partial x^{r_i^j}}\right), \tag{1}$$

$p, n_i \in N, n_i^j \in N_0, mu_i \in R$.

The similar problem for the Korteweg-de Vries equation is treated in paper [1]. In this paper we define some special sets of transformations with which searching of possible forms of P.C.L. is also presented.

In that case, when $p = 1$ or the equation (1) is linear, all equations which have at least one P.C.L. are singled out and the corresponding types of P.C.L. are written.

The suitable program code for the searching of P.C.L. is also developed.

§ 1 . DEFINITIONS AND DESIGNATIONS

Let W be set of operators L.

Let us note that P.C.L. has the following form:

$$\frac{\partial}{\partial t} S[u] + \frac{\partial}{\partial x} X[u] = 0 \tag{2}$$

where $S, X \in W$, S is the density and X is the flux [1].

Let us denote the product $\frac{\partial^{k_1} u}{\partial x^{k_1}} \frac{\partial^{k_2} u}{\partial x^{k_2}} \dots \frac{\partial^{k_n} u}{\partial x^{k_n}}$ by $[k_1, k_2, ..., k_n][u]$ and the transformation $[k_1, k_2, ..., k_n][\cdot]$ by $[k_1, k_2, ..., k_n]$.

Below the following four sets of transformations will be used:

$$V = \{T = [k_1, k_2, ..., k_n] | k_i \in N_0, n \in N, k_i \geq k_{i+1}, i = \overline{1, n-1}\},$$
$$\bar{V} = \{\bar{T} = [k_1, k_2, ..., k_n] | \bar{T} \in V, n \geq 2, k_1 = k_2\} \bigcup \{[0]\},$$

$$W = \left\{ S = \sum_{i=1}^{n} \lambda_i T_i | T_i \in V \right\}, \tag{3}$$

$$\bar{W} = \left\{ S = \sum_{i=1}^{n} \lambda_i \bar{T}_i | \bar{T}_i \in \bar{V} \right\}.$$

Let us define several special operations and relations on these sets. For example, we consider $S_1 \sim S_2$ if exists such S_3, that $S_1 - S_2 = \frac{\partial}{\partial x} S_3$. We shall say that $S_1, S_2, ..., S_n$ are linear independent, if the equivalence $\lambda_1 S_1 + \lambda_2 S_2 + ... + \lambda_n S_n \sim 0$ holds only in that case when each $\lambda_i = 0$.

It turned out that $\bar{V}$ is the basis of W and for each $S \in W$ it may be represented in only one way as the following sum:

$$S[u] = \bar{S}[u] + \frac{\partial}{\partial x} S'[u] \quad where \bar{S} \in \bar{W}, S' \in W. \tag{4}$$

Let $T = [k_1, k_2, ..., k_n]$, $R = [r_1, r_2, ..., r_n]$.

We shall assure that $T > R$ if:

$1) n > m$,

$2) n = m, t_1 = r_1, t_2 = t_2, ..., t_{i-1} = r_{i-1}, t_i > r_i$.

We shall call the density $\dot{S}$ of the equation (1) as the supporting density of (1) if:

$1) \dot{S} \in \bar{W}$,

2) The coefficient of the greatest term of $\dot{S}$ is 1,

3) $\dot{S}$ does not contain the subsum which is the density.

It is not difficult to see, that each density may be represented as a following sum:

$$S[u] = \sum_{i=1}^{M} \lambda_i \dot{S}_i[u] + \frac{\partial}{\partial x} S'[u] \tag{5}$$

where $\dot{S}_i[u], i = \overline{1, M}$ are support densities and $S' \in W$.

The representation (5) shows that it is sufficient to find the support densities only, because the whole spectrum of support densities we derive with the sum of linear combinations of support densities and $\frac{\partial}{\partial x} S'$, where $S' \in W$.

§ 2 . METHOD OF UNDETERMINED COEFFICIENTS

Let $\bar{S} = \sum_{i=1}^{N} \lambda_i \bar{T}_i$.

In this chapter we present the algorithm, which allows to find the whole spectrum of the coefficients $\{\lambda_i\}_i^N = 1$ for which $S[u]$ is a conservation density for the equation (1).

On the first stage we are defining the form of $\frac{\partial^k L}{\partial x^k} \in W, k = \overline{1, M}$, where M is the highest order of derivatives which take part in representation of S in (5).

If $L = \sum_{i=1}^{p} \mu_i [R_1^i, R_2^i, ..., R_n^i]$, then it could be proved that

$$\frac{\partial^k}{\partial x^k} L = \sum_{i=1}^{N} \sum_{j_1+j_2+...+j_{n i}=k} \mu_j C[j_1, j_2, ..., j_{ni}][R_1^i + J_1, R_2^i + J_2, ..., R_n^i + J_{ni}]$$

where

$$C(j_1, j_2, ..., j_{n_i}) = \frac{(j_1 + j_2 + ... + j_{n_i})!}{j_1! j_2! ... j_{n_i}!}. \tag{6}$$

On the second stage the derivatives with respect to t are excluded from $\frac{\partial}{\partial t} T_i u$. It is evident, that according to (1)

$$\frac{\partial^{k+1} u}{\partial x^k \partial t} = -\frac{\partial^k}{\partial x^k} Lu. \tag{7}$$

Consiquently equality holds:

$$\frac{\partial}{\partial t}[k_1, k_2, ..., k_n][u] = \sum_{i=1}^{n} [k_1, k_2, ..., k_{i-1}][u] \frac{\partial^{k_i} L[u]}{\partial t^{k_i}} [k_{i+1}, ..., k_n][u] \tag{8}$$

Due to formulae (7) and (8) we can find the required form of $\frac{\partial}{\partial t} T_i$.

The third stage:

Each $\frac{\partial}{\partial t} T_i[u]$ we present in the following form:

$$\frac{\partial}{\partial t} T_i[u] = \bar{A}_i[u] + \frac{\partial}{\partial x} A'[u] \ \ where \ \bar{A}_i \in \bar{W}, \ \ A'_i \in W.$$

After it we solve the equation:

$$\sum_{i=1}^{N} \lambda_i \bar{A}_i = 0. \tag{9}$$

Let us number all terms, which take part in (9). Let these terms be $E_1, E_2, ..., E_q$. In this temprorary basis to each $\bar{A}_i$ corresponds coordinates $(\eta_1^i, \eta_2^i, ..., \eta_q^i)$. It is easy to see, that solving of equation (9) is equivalent to solving of the following system of linear algebraic equations:

$$\sum_{i=1}^{N} \lambda_i, \eta_j^i = 0 \ \ j = \overline{1, q}. \tag{10}$$

After solving of this system we obtain the necessary spectrum of $\{\lambda_i\}_{i=1}^{N}$.

§ 3 . CONCEPT OF RANK AND ASSOCIATED RESULTS

On the set V let us define the rank as follows:

$$rank[k_1, k_2, ..., k_n] = \left(n, \sum_{i=1}^{n} k_i \right). \tag{11}$$

We shall call n as a degree of $[k_1, k_2, ..., k_n]$ term.

It turned out that the concept of rank is the fundamental concept for P.C.L. and it has many interesting properties, but we do not stop on them. Let us remark only that

1. If rank $T = (n, m)$, then the rank of each term in representing of $\frac{\partial^k}{\partial x^k} T$ is $(n, m + k)$ and

2. when we represent a term as a sum of linear combination of basis terms, the rank of each basis term equals to the rank of the represented term.

By means of the concept of rank let us define a projector in the following way:

$$P_n^m\left(\sum_{i=1}^{N}\lambda_i T_i\right)=\sum_{i=1}^{N}\lambda_{i_k}T_{i_k}, \tag{12}$$

where $rankT_{i_k} = (n,m)$ for each k and the rank of every other term in sum $\sum_{i=1}^{N}\lambda_i T_i$ is different from (n,m).

Let us remark, that the equivalence for $P_n^m(S)\sim 0, \forall n\in N, \forall m\in N_0$ is necessary and sufficient for equivalence $S\sim 0$.

Theorem: $S[u]$ *is the conservation density of* $\frac{\partial u}{\partial t}+Lu=0$ *if and only if* $L[u]\cdot J(S[u])\sim 0$, *where* J *is the Euler operator.*

By the way, if rank $[p_1,p_2,...,p_n] = (n,m)$ than $rankJ[p_1,p_2,...,p_n]=(n-1,m)$.

Let a and b be the least and the largest among the first components of rank of the terms which takes part in representation of L, and c,d - among the second components. Similarly $\bar a,\bar b$ and $\bar c,\bar d$ be the same ones, but for S. Then the next representations hold:

$$L=P_a^c(L)+P_a^{c+1}(L)+...+P_a^d(L)+$$

$$+P_{a+1}^c(L)+P_{a+1}^{c+1}(L)+...+P_{a+1}^d(L)+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+P_b^c(L)+P_b^{c+1}(L)+...+P_b^d(L), \tag{13}$$

$$S=P_{\bar a}^{\bar c}(S)+P_{\bar a}^{\bar c+1}(S)+P_{\bar a}^{\bar d}(S)+$$

$$+P_{\bar a+1}^{\bar c}(S)+P_{\bar a+1}^{\bar c+1}(S)+...+P_{\bar a+1}^{\bar d}(S)+$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+P_{\bar b}^{\bar c}(S)+P_{\bar b}^{\bar c+1}(S)+...+P_{\bar b}^{\bar d}(S).$$

$L\cdot J(S)\sim 0$ equivalence is correct if

$$P_c^a(L)\cdot J(P_{\bar a}^{\bar c}(S))\sim 0$$

$$P_d^a(L)\cdot J(P_{\bar a}^{d}(S))\sim 0$$

$$P_c^b(L)\cdot J(P_{\bar b}^{\bar c}(S))\sim 0 \tag{14}$$

$$P_d^b(L)\cdot J(P_{\bar b}^{d}(S))\sim 0,$$

i.d. it is necessary, that the "corner" terms of conservation densities were densities for the following smashed to pieces equations:

$$\frac{\partial u}{\partial t} + P^a_c(L) = 0,$$
$$\frac{\partial u}{\partial t} + P^a_d(L) = 0, \qquad (15)$$
$$\frac{\partial u}{\partial t} + P^b_c(L) = 0,$$
$$\frac{\partial u}{\partial t} + P^b_d(L) = 0.$$

Even if at least one of these equations has not any P.C.L, then the equation (1) has not any P.C.L. also.

We shall call $\frac{\partial u}{\partial t} + \tilde{L}[u] = 0$ equation as a support equation, if the ranks of each term in representation of $\tilde{L}[u]$ are equal.

It is obvious, that the equations (15) are support equations. So it is especially needed to investigate the support equations.

Let $\frac{\partial u}{\partial t} + \tilde{L}[u] = 0$ be a support equation and rank of each term be (N, M). If $rankT = (n, m)$, then the rank of each term in right hand of equality $\frac{\partial T}{\partial t} = A + \frac{\partial A'}{\partial t}$ is equal to $(N + n - 1, M + m)$.

Corrolary. *The densities of supporting equations are distributed among the linear combinations of terms which have the same rank.*

§ 4 . ON SUPPORTING DENSITIES OF SOME EVOLUTION EQUATIONS

As we have one-to-one correspondence between densities and corresponding P.C.L, the searching of densities is equivalent to searchig of P.C.L.

Let us note, that the unique first order supporting density is u and the supporting densities of second degree have the following form: $\tilde{S} = \sum_{i=1}^{N} \lambda_i \left(\frac{\partial^{k_i} u}{\partial x^{k_i}}\right)^2$.

Let us consider the linear evolution equation:

$$\frac{\partial u}{\partial t} + \mu_1 \frac{\partial^{p_1} u}{\partial x^{p_1}} + \mu_2 \frac{\partial^{p_2} u}{\partial x^{p_2}} + \ldots + \mu_M \frac{\partial^{p_M} u}{\partial x^{p_M}} = 0, \qquad (16)$$

where $p_i \leq p_{i+1}$ $i = 1, \overline{M-1}$, $\mu_i \neq 0$, $i = \overline{1, M}$.

It is clear, that u is the supporting density of the equation (16) if and only if $p_1 > 0$.

It turned out that if at least one of p_i is even then the equation (16) has no any P.C.L. of second degree, but if each of p_i is odd, then $\left(\frac{\partial^k u}{\partial x^k}\right)^2$ is the supporting density for each $k \in N_0$.

If Lu consists of only one component (i.d. $p = 1$), then the equation (1) we shall call two-component equation. It turned out to be, that those two-component equations which have at least one density distributed are among the equations of the following forms:

$$\frac{\partial u}{\partial t} + \lambda \left(\frac{\partial^k u}{\partial x^k}\right)^N \cdot \frac{\partial^{k+1} u}{\partial x^{k+1}} = 0, \quad (i)$$

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u^{p_1}}{\partial x^{p_1}} \frac{\partial u^{p_2}}{\partial x^{p_2}} = 0, \quad (ii) \tag{17}$$

$$\frac{\partial u}{\partial t} + \eta \left(\frac{\partial^q u}{\partial x^q} \right)^M = 0, \quad (iii)$$

and with amount of support densitsies we have following classification:

Two-component equations, which have infinitly many supporting densities:

1) $i : k = 0, N = 0$.

Then the supporting density is each $\frac{\partial^{k_1} u}{\partial x^{k_1}} \frac{\partial^{k_2} u}{\partial x^{k_2}} \dots \frac{\partial^{k_n} u}{\partial x^{k_n}}$, where $k_{i+1} \geq k_i; i = \overline{1,n}, k_1 = k_2$.

2) $i : k = 0, N > 0$.

Then the supporting density is $u^n, \forall n \in N$,

3) $i : k > 0, k$ is even, $N = 0$.

Then the supporting density is $\frac{\partial^j u}{\partial x^j}, \forall j \in N$.

Two-component equations, which have two supporting densities:

4) $i: k > 0, k$ is even, $N > 0$.

Then the suppotting densities are $u, \left(\frac{\partial^{\frac{k}{2}} u}{\partial x^{\frac{k}{2}}} \right)^2$.

Two-component equations, which have only one density:

5) $i : k$ is odd. Then the supporting density is u.

6) $ii : p_1 + p_2$ is odd. Then the supporting density is u.

7) $iii : M > 0, q$ is odd. Then the supporting density is $\left(\frac{\partial^{\frac{q+1}{2}} u}{\partial x^{\frac{q+1}{2}}} \right)^2$.

All other two-component equations, which are not comprised by the above cases 1)-7) have no any support density i.d. any P.C.L.

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ზოგიერთი ევოლუციური განტოლების პოლინომიალური შენახვის კანონების შესახებ

რ. სხირტლაძე

რეზიუმე

ნაშრომში მოყვანილია განსაზღვრული ტიპის ევოლუციური განტოლებების (ევ) შესაბამისი პოლინომიალური შენახვის კანონების (პშკ) ძიების მეთოდები. მოყვანილია ალგორითმი, რომელიც რიგ შემთხვევებში საშუალებას გვაძლევს ცხადი სახით გამოვწეროთ პშკ. სრულადაა შესწავლილი შემთხვევა, როცა განხილული ტიპის ევ ორი შესაკრებისაგან

შედგება, ანუ ამ შემთხვევისათვის მოყვანილია ყველა ეგ, რომელსაც ერთი მაინც პშკ აქვს და ამოწერილია ყველა შესაბამისი პშკ.

О полиномиальных законах сохранения некоторых эволюционных уравнений

Р. Схиртладзе

Резюме

В работе приведены методы нахождения полиномиальных законов сохранения (ПЗС) для эволюционных уравнений (ЭУ). Приведен алгоритм, по которому в ряде случаев можно выписать ПЗС в явной форме. Полностью исследован случай, когда ЭУ рассуждаемого типа состоит из двух слагаемых, т. е. в этом случае приведены все ЭУ, которые имеют хоть один ПЗС со всеми соответствующими ПЗС.